

IX. *On the General Resolution of Algebraical Equations.*  
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**I**N the year 1757 I sent some papers to the Royal Society, which papers were printed in the year 1759, and copies of them delivered to several persons; these papers somewhat corrected, with the addition of a second part on the properties of curve lines, were published in the year 1762. In the years 1767, 1768 and 1769 I printed, and published in the beginning of the year 1770, the same papers with additions and emendations under the title of *Meditationes Algebraicæ*. In these papers were contained, with many other inventions, the most general resolution of algebraical equations known, as it contains the resolution of every algebraical equation, of which the general resolution has been given, viz. the resolution of quadratic, cubic and biquadratic, the resolution of Mr. DE MOIVRE's and Mr. BEROUT's (since published) equations; it discovers the  
 resolution

resolution of an equation of  $n$  dimensions, of which the  $n$  roots are given, and also deduces innumerable equations of  $n$  dimensions, which contain  $n - 1$  independent coefficients. From whence it seems probable, that this new method of mine may contain the most general resolution of algebraical equations that ever has, or, perhaps, ever will be invented.

The general resolution is  $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4} \dots + r\sqrt[n]{p^{n-3}} + s\sqrt[n]{p^{n-2}} + t\sqrt[n]{p^{n-1}} + \frac{A}{n}$ , if the equation be  $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$ .

I shall add the resolution of some particular equations from this method, and then subjoin the equation to which  $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + \&c.$  is the general resolution.

1. Let the resolution be  $x = a\sqrt[3]{p} + b\sqrt[3]{p^2}$ , and the correspondent equation free from radicals will be found  $x^3 - 3abpx - a^3p - b^3p^2 = 0$ . Let  $x^3 - px - Q = 0$  be a cubic equation whose resolution is required, which suppose the same as the equation found above, and consequently their correspondent terms equal, i. e.  $p = 3abp$  and  $Q = a^3p + b^3p^2$ , whence  $p = \frac{P}{3ab}$ , which value being substituted for  $p$  in the second equation, there results  $Q = \frac{Pa^2}{3b} + \frac{b^2P^2}{9a^2}$ . In this equation for  $a$  or  $b$  may be assumed unity, or any other

other quantity whatever, and there will result an equation of the formula of a quadratic from which the other  $b$  or  $a$  may be found, whence from the equation ( $p = \frac{P}{3ab}$ )  $p$  may be deduced, and consequently the resolution of the cubic required.

In the same manner for  $p$  may be assumed any quantity whatever, and in the equation  $Q = a^3p + b^3p^2$  for  $b$  substitute its value  $\frac{P}{3ap}$ , or for  $a$  its value  $\frac{P}{3bp}$ , and there result the equations  $Q = a^3p + \frac{P^3}{27a^3p}$ , and  $Q = \frac{P^3}{27b^3p} + b^3p^2$ , which have the formula of a quadratic, from which may be deduced the resolution of the cubic required.

2. Let the resolution assumed be  $x = a\sqrt[4]{p} + b\sqrt[4]{p^2} + c\sqrt[4]{p^3}$ ; exterminate the irrational quantities, and there results the equation  $x^4 - (2b^2 + 4ac)px^2 - 4(a^2bp + bc^2p^2)x - a^4p + b^4p^2 - c^4p^3 + 2a^2c^2p^2 - 4ab^2cp^2 = 0$ ; suppose  $p = 1$ , and the given equation  $x^4 + qx^2 - rx + s = 0$ , let the correspondent terms of the given and resulting equations be respectively made equal to each other, and there result the three equations  $2b^2 + 4ac = -q$ ,  $4b(a^2 + c^2) = r$ , and  $a^4 - b^4 + c^4 - 2a^2c^2 + 4ab^2c = -s$ ; reduce these equations into one, so that the unknown quantities  $a$  and  $c$  may be exterminated, and there results the equation  $4b^6 + 2qb^4 + (\frac{r^2}{4} - s)b^2 - \frac{r^3}{16} = 0$  of the formula of a cubic, from which

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the unknown quantity  $b$  may be found, which being substituted for its value ( $b$ ) in the preceding equations, from the equations thence ensuing may be found the unknown quantities  $a$  and  $c$ , and consequently the resolution of the given biquadratic  $x^4 + qx^2 - rx + s = 0$ .

From the same principles can be deduced different resolutions of the above-mentioned biquadratic  $x^4 + qx^2 - rx + s = 0$ .

3. I. Let  $x = a\sqrt[n]{p} + b\sqrt[n]{p^2}$ , then will the equation free from radicals be  $x^{2n} - 2b^n p x^n - \frac{n}{1,2} \cdot 2nb^{n-1} a^2 p x^{n-1} - \frac{n \times n^2 - 1}{1,2,3,4} \times 2nb^{n-2} a^4 p x^{n-2} - \frac{n \times n^2 - 1 \times n^2 - 4}{1,2,3,4,5,6} \times 2nb^{n-3} a^6 p x^{n-3} - \frac{n \cdot n^2 - 1 \cdot n^2 - 4 \cdot n^2 - 9}{1,2,3,4,5,6,7,8} \times 2nb^{n-4} a^8 p x^{n-4} \dots - \frac{n \cdot n^2 - 1 \times n^2 - 4 \times n^2 - 9 \times n^2 - 16 \dots n^2 - n^2 - 2}{1,2,3,4,5,6,7 \dots 2n-2} \times 2na^{2n-2} b p x = a^n p - b^n p^2$ .

This equation may be deduced from the following principles. Let  $\alpha, \beta, \gamma, \delta, \epsilon$ , &c. be the  $2n$  roots of the equation  $x^{2n} - 1 = 0$ , then (by Prop. XXIII. of my Meditat. Algebraicæ) the equation free from radicals will be the product of the following quantities  $(x - a\alpha\sqrt[n]{p} - b\alpha^2\sqrt[n]{p^2}) (x - a\beta\sqrt[n]{p} - b\beta^2\sqrt[n]{p^2}) (x - a\gamma\sqrt[n]{p} - b\gamma^2\sqrt[n]{p^2}) (x - a\delta\sqrt[n]{p} - b\delta^2\sqrt[n]{p^2}) (x - a\epsilon\sqrt[n]{p} - b\epsilon^2\sqrt[n]{p^2})$  &c.  $= 0$ : multiply these quantities into each other, and from the resulting product, by Prob. III. of the Meditat. Algebr. easily can

be deduced the equation free from radicals which was to be found.

3. II. Let  $x = a\sqrt[n]{p} + b\sqrt[n]{p^2}$ , then will the correspondent equation free from radicals be  $x^{2n+1} - \overline{2n+1} b^n a p x^n - \frac{n \cdot n+1}{1 \cdot 2 \cdot 3} \times \overline{2n+1} b^{n-1} a^3 p x^{n-1} - \frac{n \times n^2 - 1 \times n + 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \overline{2n+1} b^{n-2} a^5 p x^{n-2} - \frac{n \times n^2 - 1 \times n^2 - 4 \times n + 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \times \overline{2n+1} b^{n-3} a^7 p x^{n-3} - \frac{n \cdot n^2 - 1 \cdot n^2 - 4 \cdot n^2 - 9 \cdot n + 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \times \overline{2n+1} b^{n-4} a^9 p x^{n-4} - \dots - \frac{n \cdot n^2 - 1 \cdot n^2 - 4 \cdot n^2 - 9 \cdot n^2 - 16 \dots n^2 - n - 2^2 \times 2n-1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \dots 2n-1} \times \overline{2n+1} b a^{2n-1} p x = a^{2n+1} p + b^{2n+1} p^2.$

This may be derived from the same principles as the preceding.

3. III. In general let the equation be  $x = a\sqrt[m]{p} + b\sqrt[m]{p^2}$ , then will the equation free from radicals become  $x^m - m a^{m-2} b p x - m \cdot \frac{m-3}{2} a^{m-4} b^2 p x^2 - m \cdot \frac{m-4}{2} \cdot \frac{m-5}{3} a^{m-6} b^3 p x^3 - m \cdot \frac{m-5}{2} \cdot \frac{m-6}{3} \cdot \frac{m-7}{4} a^{m-8} b^4 p x^4 - m \cdot \frac{m-6}{2} \cdot \frac{m-7}{3} \cdot \frac{m-8}{4} \cdot \frac{m-9}{5} a^{m-10} b^5 p x^5 - \&c. = a^m p \pm b^m p^2$ ; if  $m$  denotes an even number, it will be  $-b^m p^2$ , but if an odd number, it will be  $+b^m p^2$ .

4. I. Let  $n$  denote an odd number, and  $x = a\sqrt[n]{p} + b\sqrt[n]{p^2}$ , then will  $x^n - p (n a^{n-3} b x^2 + n \cdot \frac{n-5}{2} a^{n-6} b^2 x^4 + n \cdot \frac{n-7}{2} a^{n-9} b^3 x^5 + n \cdot \frac{n-9}{2} \cdot \frac{n-10}{3} \cdot \frac{n-11}{4} a^{n-12} b^4 x^8 + n \cdot \frac{n-11}{2} \cdot \frac{n-12}{3} \cdot \frac{n-13}{4} a^{n-14} b^5 x^{10} + n \cdot \frac{n-13}{2} \cdot \frac{n-14}{3} \cdot \frac{n-15}{4} \cdot \frac{n-16}{5} \cdot \frac{n-17}{6} a^{n-18} b^6 x^{12} + n \cdot \dots)$

$$\begin{aligned}
 & \frac{n-15}{2} \cdot \frac{n-16}{3} \cdot \frac{n-17}{4} \cdot \frac{n-18}{5} \cdot \frac{n-19}{6} \cdot \frac{n-20}{7} a^{\frac{n-21}{2}} b^7 x^{14} + \&c.) \pm p^2 (na^{\frac{n-3}{2}} \\
 & b^{\frac{n+1}{2}} x - \frac{1}{2^2} \times n \cdot \frac{n-5}{2} \cdot \frac{n-7}{3} a^{\frac{n-9}{2}} b^{\frac{n+3}{2}} x^3 + \frac{1}{2^4} \times n \cdot \frac{n-7}{2} \cdot \frac{n-9}{3} \cdot \frac{n-11}{4} \cdot \frac{n-13}{5} \\
 & a^{\frac{n-15}{2}} b^{\frac{n+5}{2}} x^5 - \frac{1}{2^6} \times n \cdot \frac{n-9}{2} \cdot \frac{n-11}{3} \cdot \frac{n-13}{4} \cdot \frac{n-15}{5} \cdot \frac{n-17}{6} \cdot \frac{n-19}{7} a^{\frac{n-21}{2}} b^{\frac{n+7}{2}} x^7 + \\
 & \frac{1}{2^8} \times n \cdot \frac{n-11}{2} \cdot \frac{n-13}{3} \cdot \frac{n-15}{4} \cdot \frac{n-17}{5} \cdot \frac{n-19}{6} \cdot \frac{n-21}{7} \cdot \frac{n-23}{8} \cdot \frac{n-25}{9} a^{\frac{n-27}{2}} b^{\frac{n+9}{2}} x^9 - \\
 & \&c.) = a^n p + b^n p^3.
 \end{aligned}$$

The quantity  $\pm p^{\frac{n-3}{4}}$  denotes  $+p^2$  if  $\frac{n-3}{4}$  is a whole number, otherwise  $-p^2$ .

4. II. Let  $n$  denote an even number, and  $x$  as before  
 $= a\sqrt[n]{p} + b\sqrt[n]{p^3}$ , then will  $x^n - p (na^{n-3}bx^2 + n \cdot \frac{n-5}{2} a^{n-6}b^2x^4$   
 $+ n \cdot \frac{n-7}{2} \cdot \frac{n-8}{3} a^{n-9}b^3x^6 + n \cdot \frac{n-9}{2} \cdot \frac{n-10}{3} \cdot \frac{n-11}{4} a^{n-12}b^4x^8 + n \cdot \frac{n-11}{2}$   
 $\cdot \frac{n-12}{3} \cdot \frac{n-13}{4} \cdot \frac{n-14}{5} a^{n-15}b^5x^{10} + \&c.) \pm p^2 (\frac{1}{2}n \cdot \frac{n-4}{2} a^{\frac{n-6}{2}} b^{\frac{n+2}{2}} x^2 -$   
 $\frac{1}{2^3} \times n \cdot \frac{n-6}{2} \cdot \frac{n-8}{3} \cdot \frac{n-10}{4} a^{\frac{n-12}{2}} b^{\frac{n+4}{2}} x^4 + \frac{1}{2^5} \times n \cdot \frac{n-8}{2} \cdot \frac{n-10}{3} \cdot \frac{n-12}{4} \cdot \frac{n-14}{5}$   
 $\cdot \frac{n-16}{6} a^{\frac{n-18}{2}} b^{\frac{n+6}{2}} x^6 - \frac{1}{2^7} \times n \cdot \frac{n-10}{2} \cdot \frac{n-12}{3} \cdot \frac{n-14}{4} \cdot \frac{n-16}{5} \cdot \frac{n-18}{6} \cdot \frac{n-20}{7} \cdot \frac{n-22}{8}$   
 $a^{\frac{n-24}{2}} b^{\frac{n+8}{2}} x^8 + \&c.) = a^n p + b^n p^3 \pm 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^2.$

The quantities  $\pm p^2$  and  $\pm 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$  denote  $+p^2$ , and  
 $+2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$ , if  $\frac{n-2}{4}$  is a whole number, otherwise they  
 denote  $-p^2$  and  $-2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$  respectively.

5. I. Let  $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$ , and  $n$  an odd number,

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then

then will  $x^n - nabpx^{n-3} + n \cdot \frac{n-3}{2} a^2 b^2 p^2 x^{n-4} - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3}$

$$a^3 b^3 p^3 x^{n-6} + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} a^4 b^4 p^4 x^{n-8} - \&c. = a^n p + b^n p^{n-1}.$$

5. II. Let  $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$ , and  $n$  an even number,

then will  $x^n - nabpx^{n-2} + n \cdot \frac{n-3}{2} a^2 b^2 p^2 x^{n-4} - n \cdot \frac{n-4}{2}$

$$\frac{n-5}{3} a^3 b^3 p^3 x^{n-6} + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} a^4 b^4 p^4 x^{n-8} - \&c. = a^n p +$$

$$2a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} + b^n p^{n-1}; \text{ it will be } + 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} \text{ if } n = 4r + 2;$$

$$\text{but } - 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} \text{ if } n = 4r.$$

6. I. Let  $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-2}}$ , and  $n$  an odd number, which has not the number 3 for a divisor, then will

$$x^n - na^2 b p x^{n-3} + n \cdot \frac{n-5}{2} a^4 b^2 p^2 x^{n-6} - n \cdot \frac{n-7}{2} \cdot \frac{n-8}{3} a^6 b^3 p^3 x^{n-9} +$$

$$n \cdot \frac{n-9}{2} \cdot \frac{n-10}{3} \cdot \frac{n-11}{4} a^8 b^4 p^4 x^{n-12} - n \cdot \frac{n-11}{2} \cdot \frac{n-12}{3} \cdot \frac{n-13}{4} \cdot \frac{n-14}{5}$$

$$a^{10} b^5 p^5 x^{n-15} + n \cdot \frac{n-13}{2} \cdot \frac{n-14}{3} \cdot \frac{n-15}{4} \cdot \frac{n-16}{5} \cdot \frac{n-17}{6} a^{12} b^6 p^6 x^{n-18} - \&c.$$

(to  $m$  terms, where  $m$  is the number either equal to, or

$$\text{the least greater than } \frac{n}{3}) - ab^{\frac{n+1}{2}} p^{\frac{n-1}{2}} (nx^{\frac{n-3}{2}} + \frac{1}{4}n \cdot \frac{n-5}{2} \cdot \frac{n-7}{3}$$

$$a^2 b p x^{\frac{n-9}{2}} + \frac{1}{2}n \cdot \frac{n-7}{2} \cdot \frac{n-9}{3} \cdot \frac{n-11}{4} \cdot \frac{n-13}{5} a^4 b^2 p^2 x^{\frac{n-15}{2}} + \frac{1}{2} \times n \cdot \frac{n-9}{2}$$

$$\frac{n-11}{3} \cdot \frac{n-13}{4} \cdot \frac{n-15}{5} \cdot \frac{n-17}{6} \cdot \frac{n-19}{7} a^6 b^3 p^3 x^{\frac{n-21}{2}} + \frac{1}{2} \times n \cdot \frac{n-11}{2} \cdot \frac{n-13}{3} \cdot \frac{n-15}{4}$$

$$\frac{n-17}{5} \cdot \frac{n-19}{6} \cdot \frac{n-21}{7} \cdot \frac{n-23}{8} \cdot \frac{n-25}{9} a^8 b^4 p^4 x^{\frac{n-27}{2}} + \&c.) = A = a^n p +$$

$$b^n p^{n-2}.$$

Let  $n$  be an odd number divisible by 3, then will the

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above-mentioned quantity =  $A = a^n p + b^n p^{n-2} + 3a^{\frac{2n}{3}} b^{\frac{2n}{3}} p^{\frac{n}{3}-1} + 3a^{\frac{2n}{3}} b^{\frac{n}{3}} p^{\frac{n}{3}}$ .

6. II. Let  $n$  be an even number, not divisible by 3, then will  $x^n - na^2 b p x^{n-3} + n \cdot \frac{n-5}{2} a^4 b^2 p^2 x^{n-6} - n \cdot \frac{n-7}{2} \cdot \frac{n-8}{3} a^6 b^3 p^3 x^{n-9} + n \cdot \frac{n-9}{2} \cdot \frac{n-10}{3} \cdot \frac{n-11}{4} a^8 b^4 p^4 x^{n-12} - n \cdot \frac{n-11}{2} \cdot \frac{n-12}{3} \cdot \frac{n-13}{4} \cdot \frac{n-14}{5} a^{10} b^5 p^5 x^{n-15} + n \cdot \frac{n-13}{2} \cdot \frac{n-14}{3} \cdot \frac{n-15}{4} \cdot \frac{n-16}{5} \cdot \frac{n-17}{6} a^{12} b^6 p^6 x^{n-18} - \&c.$  to  $m$  terms as before  $- b^{\frac{n}{2}} p^{\frac{n}{2}-1} (2x^{\frac{n}{2}} + \frac{1}{2} n \cdot \frac{n-4}{2} a^2 b p x^{\frac{n}{2}-3} + \frac{1}{2^3} \times n \cdot \frac{n-6}{2} \cdot \frac{n-8}{3} \cdot \frac{n-10}{4} a^4 b^2 p^2 x^{\frac{n}{2}-6} + \frac{1}{2^5} \times n \cdot \frac{n-8}{2} \cdot \frac{n-10}{3} \cdot \frac{n-12}{4} \cdot \frac{n-14}{5} \cdot \frac{n-16}{6} a^6 b^3 p^3 x^{\frac{n}{2}-9} + \frac{1}{2^7} \times n \cdot \frac{n-10}{2} \cdot \frac{n-12}{3} \cdot \frac{n-14}{4} \cdot \frac{n-16}{5} \cdot \frac{n-18}{6} \cdot \frac{n-20}{7} \cdot \frac{n-22}{8} a^8 b^4 p^4 x^{\frac{n}{2}-12} + \&c.) = A = a^n p - b^n p^{n-2}$ .

Let  $n$  be an even number divisible by 3; then will the above-mentioned quantity  $A = a^n p - b^n p^{n-2} - 3a^{\frac{2n}{3}} b^{\frac{n}{3}} p^{\frac{n}{3}-1} + 3a^{\frac{2n}{3}} b^{\frac{2n}{3}} p^{\frac{n}{3}}$ .

In all the preceding cases  $n$ ,  $m$  and  $r$  denote whole affirmative numbers.

These equations may be deduced in the same manner as is before given in Case 3. I; or can be demonstrated by writing in the equation free from radicals for the different powers of  $x$  their values deduced from the given equation  $x = a\sqrt[r]{p} + b\sqrt[r]{p}$ .

To



To render the solution general, it may not be improper to subjoin the subsequent.

## L E M M A.

1. Let  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \&c.$  be the respective roots of the equation  $x^n - 1 = 0$ ; then will  $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m + \zeta^m + \&c. = 0$ , unless  $n = m$ , or  $n$  is a divisor of  $m$ , in which case  $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m + \zeta^m + \&c. = n$ .

2. The sum of all quantities of the following kind  $\alpha^m \beta^r + \alpha^r \beta^m + \alpha^m \gamma^r + \alpha^r \gamma^m + \beta^m \gamma^r + \beta^r \gamma^m + \alpha^m \delta^r + \&c.$  will be  $= 0$ ; unless  $n$  be either equal to, or a divisor of  $m + r$ , in which case the sum above-mentioned will be  $= -n$ ; except  $n$  be either equal to  $m$  or  $r$ , or a divisor of them, in which case the sum will be  $n^2 - n$ ; but if  $m = r$ , then in the former case will the above-mentioned sum  $= -\frac{n}{2}$ , and in the latter  $= \frac{n^2 - n}{2}$ .

3. The sum of all quantities of this kind  $\alpha^m \beta^r \gamma^s \delta^t + \alpha^r \beta^m \gamma^s \delta^t + \alpha^m \beta^r \gamma^t \delta^s + \alpha^r \beta^m \gamma^t \delta^s + \alpha^m \beta^r \gamma^s \delta^t + \&c.$  will be  $= 0$ , unless  $n$  be either equal to  $r + m + s + t + \&c.$  or a divisor of it.

Let  $\pi$  be the number of indices  $m, r, s, t, \&c.$  and  $n$  be either equal to  $m + r + s + t + \&c.$  or a divisor of it, but  $n$  be neither equal to, nor a divisor of the sum of any two, three, four,  $\dots \pi - 3, \pi - 2$  or  $\pi - 1$  of the above-mentioned

mentioned quantities; then will the sum above-mentioned =  $\frac{1.2.3.4... \pi - 2. \pi - 1 \times n}{1}$ ; where it will be +, if  $\pi$  be an odd number; otherwise -.

In this case, if  $a$  indices be  $m$ ,  $b$  indices be  $r$ ,  $c$  indices be  $s$ ,  $d$  indices be  $t$ , &c. then will the above-mentioned sum =

$$\frac{1.2.3.4... \pi - 2 \times \pi - 1}{1.2.3...a \times 1.2.3...b \times 1.2.3...c \times 1.2.3...d \times \&c.} \times n.$$

Let  $n$  be either equal to, or a divisor of the sum of any number  $\rho$  (less than  $\pi$ ) of the above-mentioned quantities  $m, r, s, t$ , &c. and consequently either equal to, or a divisor of the sum of the  $(\pi - \rho)$  remaining quantities: find the sum of all possible quantities of this kind  $1.2.3... \rho - 2 \times \rho - 1 \times 1.2.3... \pi - \rho - 2 \times \pi - \rho - 1 \times n^2$ , which sum call A.

Let  $n$  be either equal to, or a divisor of the sum of any number  $(\sigma)$  of the above-mentioned quantities  $m, r, s, t$ , &c; and also equal to, or a divisor of the sum of any number  $(\rho')$  of the remaining quantities, and consequently it will be either equal to, or a divisor of, the sum of the  $(\pi - \rho' - \sigma)$  remaining quantities; then find the sum of all possible quantities of this sort  $1.2.3... \sigma - 2 \times \sigma - 1 \times 1.2.3... \rho' - 2 \times \rho' - 1 \times 1.2.3... \pi - \rho' - \sigma - 2 \times \pi - \rho' - \sigma - 1 \times n^3$ , which sum call B.

In the same manner let  $n$  be either equal to, or a divisor of the sum of any number  $(\tau)$  of the above-mentioned

tioned quantities  $m, r, s, t, \&c$ ; and similarly let  $n$  be either equal to, or a divisor of the sum of any number ( $\sigma$ ) of the remaining quantities; and also let  $n$  be either equal to, or a divisor of the sum of any number ( $\sigma''$ ) of the remaining quantities; then will  $n$  be either equal to, or a divisor of the sum of the  $(\pi - \tau - \sigma' - \rho'')$  remaining quantities: find the sum of all quantities of this sort  $1.2.3.. \tau - 2 \times \tau - 1 \times 1.2.3.. \sigma' - 2 \times \sigma' - 1 \times 1.2.3.. \rho'' - 2 \times \rho'' - 1 \times 1.2.3.. \pi - \tau - \sigma' - \rho'' - 2 \times \pi - \tau - \sigma' - \rho'' - 1 \times n^4$ , which sum call  $c$ ; and so on; then will the above-mentioned sum  $\alpha^m \beta^r \gamma^s \delta^t \&c. + \alpha^r \beta^m \gamma^s \delta^t \&c. + \alpha^m \beta^r \gamma^s \delta^s \&c. + \alpha^r \beta^s \gamma^m \delta^t \&c. + \&c. = \mp (1.2.3.. \pi - 2 \times \pi - 1 \times n - A + B - C + D - \&c.)$  where it will be  $+$  if  $\pi$  be an odd number, otherwise  $-$ .

In this case, if  $a$  indices be  $m$ ,  $b$  indices be  $r$ ,  $c$  indices be  $s$ ,  $d$  indices be  $t$ ,  $\&c$ . then will the above-mentioned sum  $= \mp \frac{1.2.3.. \pi - 2 \times \pi - 1 \times n - A + B - C + D - \&c.}{1.2.3.. a \times 1.2.3.. b \times 1.2.3.. c \times 1.2.3.. d \times \&c.}$ .

7. Let  $\alpha, \beta, \gamma, \delta, \epsilon, \&c$ . be the roots of the equation  $x^n - 1 = 0$ , and the resolution be  $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4} \dots + b\sqrt[n]{p^{\lambda}} \dots + k\sqrt[n]{p^{\mu}} \dots + l\sqrt[n]{p^{\nu}} \dots + q\sqrt[n]{p^{\xi}} \dots + r\sqrt[n]{p^{\theta}} \dots + s\sqrt[n]{p^{n-4}} + t\sqrt[n]{p^{n-3}} + v\sqrt[n]{p^{n-2}} + u\sqrt[n]{p^{n-1}}$ ; then will the different values of  $x$  be respectively  $a\sqrt[n]{p} \times \alpha + b\sqrt[n]{p^2} \times \alpha^2 + c\sqrt[n]{p^3} \times \alpha^3 + d\sqrt[n]{p^4} \times \alpha^4 \dots + b\sqrt[n]{p^{\lambda}} \times \alpha^{\lambda} \dots + k\sqrt[n]{p^{\mu}} \times \alpha^{\mu} \dots + s\sqrt[n]{p^{n-4}} \times \alpha^{n-4} + t\sqrt[n]{p^{n-3}} \times \alpha^{n-3} + v\sqrt[n]{p^{n-2}} \times \alpha^{n-2} + u\sqrt[n]{p^{n-1}} \times \alpha^{n-1}$ ;

$a\sqrt[n]{p}$

$$a\sqrt[n]{p} \times \beta + b\sqrt[n]{p^2} \times \beta^2 + c\sqrt[n]{p^3} \times \beta^3 + d\sqrt[n]{p^4} \times \beta^4 \dots + b\sqrt[n]{p^\lambda} \times \beta^\lambda \dots + k\sqrt[n]{p^\mu} \times \beta^\mu \dots + s\sqrt[n]{p^{n-4}} \times \beta^{n-4} + t\sqrt[n]{p^{n-3}} \times \beta^{n-3} + v\sqrt[n]{p^{n-2}} \times \beta^{n-2} + u\sqrt[n]{p^{n-1}} \times \beta^{n-1};$$

$$a\sqrt[n]{p} \times \gamma + b\sqrt[n]{p^2} \times \gamma^2 + c\sqrt[n]{p^3} \times \gamma^3 + d\sqrt[n]{p^4} \times \gamma^4 \dots + b\sqrt[n]{p^\lambda} \times \gamma^\lambda \dots + k\sqrt[n]{p^\mu} \times \gamma^\mu \dots + s\sqrt[n]{p^{n-4}} \times \gamma^{n-4} + t\sqrt[n]{p^{n-3}} \times \gamma^{n-3} + v\sqrt[n]{p^{n-2}} \times \gamma^{n-2} + u\sqrt[n]{p^{n-1}} \times \gamma^{n-1};$$

$$a\sqrt[n]{p} \times \delta + b\sqrt[n]{p^2} \times \delta^2 + c\sqrt[n]{p^3} \times \delta^3 + d\sqrt[n]{p^4} \times \delta^4 \dots + b\sqrt[n]{p^\lambda} \times \delta^\lambda \dots + k\sqrt[n]{p^\mu} \times \delta^\mu \dots + s\sqrt[n]{p^{n-4}} \times \delta^{n-4} + t\sqrt[n]{p^{n-3}} \times \delta^{n-3} + v\sqrt[n]{p^{n-2}} \times \delta^{n-2} + u\sqrt[n]{p^{n-1}} \times \delta^{n-1};$$

&c.

&c.

&c.

&c.

and consequently the sum of the values or roots, which is the coefficient of the second term of the equation sought, will be  $a\sqrt[n]{p} \times \alpha + \beta + \gamma + \delta + \&c. (o) + b\sqrt[n]{p^2} \times \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c. (o) + c\sqrt[n]{p^3} \times \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \&c. (o) + \dots + v\sqrt[n]{p^{n-2}} \times \alpha^{n-2} + \beta^{n-2} + \gamma^{n-2} + \delta^{n-2} + \&c. (o) + u\sqrt[n]{p^{n-1}} \times \alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + \delta^{n-1} + \&c. (o) = 0$ .

The sum of the products of every two of the values or roots, which is the coefficient of the third term of the equation sought, will be  $a^2\sqrt[n]{p^2} \times \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \&c. (o) + ab\sqrt[n]{p^3} \times \alpha\beta^2 + \beta\alpha^2 + \alpha\gamma^2 + \gamma\alpha^2 + \beta\gamma^2 + \gamma\beta^2 + \alpha\delta^2 + \delta\alpha^2 + \&c. (o)$ , and in general all the terms will be 0, unless  $a \times u \times p \times \alpha\beta^{n-1} + \beta\alpha^{n-1} + \alpha\gamma^{n-1} + \gamma\alpha^{n-1} + \beta\gamma^{n-1} +$

$$\begin{aligned}
& \gamma\beta^{n-1} + \alpha\delta^{n-1} + \delta\alpha^{n-1} + \beta\delta^{n-1} + \delta\beta^{n-1} + \gamma\delta^{n-1} + \delta\gamma^{n-1} + 8\epsilon c. (-n) \\
& + b \times v \times p \times \alpha^2\beta^{n-2} + \beta^2\alpha^{n-2} + \alpha^2\gamma^{n-2} + \gamma^2\alpha^{n-2} + \beta^2\gamma^{n-2} + \gamma^2\beta^{n-2} \\
& + \alpha^2\delta^{n-2} + \delta^2\alpha^{n-2} + \beta^2\delta^{n-2} + \delta^2\beta^{n-2} + 8\epsilon c. (-n) + ctp \times \\
& \alpha^3\beta^{n-3} + \beta^3\alpha^{n-3} + \alpha^3\gamma^{n-3} + \gamma^3\alpha^{n-3} + \beta^3\gamma^{n-3} + \gamma^3\beta^{n-3} + \alpha^3\delta^{n-3} + \\
& \delta^3\alpha^{n-3} + \beta^3\delta^{n-3} + \delta^3\beta^{n-3} + 8\epsilon c. (-n) + dsp \times \alpha^4\beta^{n-4} + \beta^4\alpha^{n-4} + \\
& \alpha^4\gamma^{n-4} + \gamma^4\alpha^{n-4} + \beta^4\gamma^{n-4} + \gamma^4\beta^{n-4} + 8\epsilon c. (-n) + 8\epsilon c. = -np(au + \\
& bv + ct + ds + 8\epsilon c.)
\end{aligned}$$

If  $n = 2\lambda$ , then will the coefficient of  $b^2p$  be  $\frac{n}{2}$ , i. e. the above-mentioned coefficient will be  $-np(au + bv + ct + ds + \dots + \frac{1}{2}b^2)$ .

The sum of the contents of every three of the above-mentioned values or roots, which is the coefficient of the fourth term of the equation required, will be  $a^3\sqrt[3]{p^3} \times \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + 8\epsilon c. (0) + a^2b\sqrt[3]{p^4} \times \alpha\beta\gamma^2 + \alpha\gamma\beta^2 + \beta\gamma\alpha^2 + \alpha\beta\delta^2 + 8\epsilon c. (0) + 8\epsilon c. + a^2v\sqrt[3]{p^4} \times \alpha\beta\gamma^{n-2} + \alpha\gamma\beta^{n-2} + \beta\gamma\alpha^{n-2} + \alpha\beta\delta^{n-2} + \alpha\delta\beta^{n-2} + 8\epsilon c. (\frac{1.2.n}{1.2}) + abt\sqrt[3]{p^n} \times \alpha\beta^2\gamma^{n-3} + \alpha\gamma^2\beta^{n-3} + \beta\alpha^2\gamma^{n-3} + \beta\gamma^2\alpha^{n-3} + \gamma\alpha^2\beta^{n-3} + \gamma\beta^2\alpha^{n-3} + \alpha\beta^2\delta^{n-3} + 8\epsilon c. (1.2.n) + 8\epsilon c$ ; and in general all the terms (unless the quantity  $\sqrt[3]{p^0}$  contained in the term have this formula  $\sqrt[3]{p^n} = p$ , or  $\sqrt[3]{p^{2n}} = p^2$ ) will be  $\propto$ ; let the general term be denoted by  $blk\sqrt[3]{p^{\lambda+\mu+\nu}}$   $\times \alpha^\lambda\beta^\mu\gamma^\nu + \alpha^\lambda\beta^\nu\gamma^\mu + \alpha^\mu\beta^\lambda\gamma^\nu + \alpha^\mu\beta^\nu\gamma^\lambda + \alpha^\nu\beta^\lambda\gamma^\mu + \alpha^\nu\beta^\mu\gamma^\lambda + \alpha^\lambda\beta^\mu\delta^\nu + 8\epsilon c$ . first let  $\lambda + \mu + \nu$  neither be equal to  $n$  or  $2n$ , then will the term

term above-mentioned = 0; if it be equal to  $n$  or  $2n$ , then will the term be  $1.2 \times n \times bklp$  or  $1.2nbklp^2$ .

If two of the three indexes  $\lambda, \mu, \nu$  be equal to each other, then divide the above-mentioned term by  $1.2$ ; if the three indexes be equal, i. e.  $\lambda = \mu = \nu$ , divide it by  $1.2.3$ : find all quantities of this kind where  $\lambda + \mu + \nu$  either is equal to  $n$  or  $2n$ , and add all the term from thence derived, and call the sum of them A.

The sum of the contents of every four of the values or roots above-mentioned, which is the coefficient of the fourth term of the equation required, will be  $a^4 \sqrt{p^4} \times \alpha\beta\gamma\delta + \alpha\beta\gamma\delta + \&c. (0) + a^3 b \sqrt{p^5} \times \alpha\beta\gamma\delta^2 + \alpha\beta\gamma^2\delta + \alpha\beta^2\gamma\delta + \alpha^2\beta\gamma\delta + \&c. (0) + \&c$ : let  $bklq \sqrt{p^{\lambda+\mu+\nu+\xi}} \times \frac{\alpha^\lambda \beta^\mu \gamma^\nu \delta^\xi + \alpha^\lambda \beta^\mu \gamma^\xi \delta^\nu + \alpha^\lambda \beta^\nu \gamma^\mu \delta^\xi + \alpha^\lambda \beta^\nu \gamma^\xi \delta^\mu + \alpha^\lambda \beta^\xi \gamma^\nu \delta^\mu + \alpha^\lambda \beta^\xi \gamma^\mu \delta^\nu + \alpha^\mu \beta^\lambda \gamma^\nu \delta^\xi + \&c.}{\text{denote a general term; this term will be}}$  = 0, unless  $\lambda + \mu + \nu + \xi$  either =  $n$  or  $2n$  or  $3n$ ; in which case the term will be either  $-1.2.3nbklqp$  or  $-1.2.3nbklqp^2$  or  $-1.2.3nbklqp^3$ ; unless  $\lambda + \mu = \nu + \xi = n$ , when the above-mentioned term will be  $-(1.2.3n - n^2) bklqp^2$ ; in this case if  $\lambda = \nu$ , and consequently  $\mu = \xi$ , then it will be  $-(1.2.3n - 1.2n^2) bklqp^2$ ; but if  $\lambda = \mu = \nu = \xi = \frac{n}{2}$ , then will the term be  $-(1.2.3n - 3n^2) bklqp^2$ .

In all these cases, if two of the indexes  $\lambda, \mu, \nu, \xi$  be equal, then must the term given above be divided by

1.2; if three, by 1.2.3; if four, by 1.2.3.4; and lastly if two are equal to each other, and the two remaining indexes equal to each other, but not to the former two, then must the term aforesaid be divided by 1.2.1.2.

Find the sum of all the possible terms of this kind, which call B.

In the same manner from the preceding Lemma may be found the aggregates of the contents of every five, six, seven, &c. roots or values multiplied into each other, which call respectively c, d, e, &c; then will the equation required be  $x^n * -np(au+bv+ct+ds+\&c.)x^{n-2} - Ax^{n-3} + Bx^{n-4} - Cx^{n-5} + Dx^{n-6} - \&c. = 0$ .

From the same principles may be deduced the most general reduction yet known of equations to others of inferior dimensions, *e. g.*

Let (X)  $x^n + (A + a\sqrt[m]{p} + b\sqrt[m]{p^2} + c\sqrt[m]{p^3} + \dots + s\sqrt[m]{p^{m-2}} + t\sqrt[m]{p^{m-1}})x^{n-1} + (B + a'\sqrt[m]{p} + b'\sqrt[m]{p^2} + \dots + s'\sqrt[m]{p^{m-2}} + t'\sqrt[m]{p^{m-1}})x^{n-2} + (C + a''\sqrt[m]{p} + b''\sqrt[m]{p^2} + \&c.)x^{n-3} + \&c. = 0$ ; let  $\alpha, \beta, \gamma, \delta, \&c.$  be the respective roots of the equation  $z^m - 1 = 0$ , then, from the principles before given, may be formed the different values of the equation X, which being multiplied into each other from the propositions before-mentioned of the *Meditationes Algebraicæ*, may be deduced an equation of  $nm$  dimensions free from radicals,

whose root is  $x$ , and which contains  $mn$  unknown quantities  $A, a, b, c, \&c. B, a', b', c', \&c. c, a'', b'', c'', \&c. p$ : for one, two or more of these unknown quantities may be assumed any quantities whatever, and thence may be deduced equations of  $mn$  dimensions, which may be reduced to equations  $x^n + (A + a\sqrt[m]{p} + b\sqrt[m]{p^2} + c\sqrt[m]{p^3} + \&c.)x^{n-1} + \&c. = 0$  of  $n$  dimensions.

In the same manner may be assumed equations, which involve  $\sqrt[m]{p}, \sqrt[m]{p^2}, \dots, \sqrt[m]{p^{m-1}}; \sqrt[3]{q}, \sqrt[3]{q^2}, \sqrt[3]{q^3}, \dots, \sqrt[3]{q^{r-1}}; \sqrt[4]{\tau}, \sqrt[4]{\tau^2}, \sqrt[4]{\tau^3}, \dots, \sqrt[4]{\tau^{s-1}}, \&c$ ; and from so reducing them as to exterminate the irrational quantities, may often be derived equations whose resolutions or reductions are known.

The method of transforming algebraical equations into others, whose roots bear any assignable algebraical (but not exponential) relation to the roots of a given algebraical equation first published by me in the papers sent to the Royal Society, and afterwards in the year 1760; and thirdly in my *Miscellanea Analytica*; and lastly in the *Meditationes Algebraicæ*, and since published by Mr. LE GRANGE in the Berlin Acts, is perhaps (as Mr. LE GRANGE observes) more general than Mr. HUDDE's, or any transformation yet invented; it is very useful in the resolution of numerous problems; and  
further



further has this peculiar advantage over all other transformations yet invented, that it often easily discovers some of the first terms of the equation required, from which many elegant Theorems may be derived.

In the works above-mentioned, viz. Miscell. Analyt. Medit. Algèb. &c. are given some problems serving to this transformation; the first of which is a series, which from the coefficients of a given algebraical equation ( $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$ ) finds the sum of any power of the roots (viz.  $\alpha^m + \beta^m + \gamma^m + \delta^m + \&c.$  where  $\alpha, \beta, \gamma, \delta, \&c.$  denote the roots of the given equation), the law of which series was published by me many years before that it was given by Mr. EULER. The third Problem often mentioned in this paper is an elegant and useful series for finding the sum of quantities of the following kind, viz.  $\alpha^m \beta^r \gamma^s \delta^t, \&c. + \alpha^r \beta^m \gamma^s \delta^t, \&c. + \alpha^s \beta^m \gamma^r \delta^t \&c. + \alpha^s \beta^m \gamma^r \delta^s \&c. + \alpha^m \beta^r \gamma^s \delta^s \&c. + \&c.$

Mr. EULER gave the following resolution,  $x = \sqrt[n]{\pi} + \sqrt[n]{\rho} + \sqrt[n]{\sigma} + \sqrt[n]{\tau} + \&c.$  where  $\pi, \rho, \sigma, \tau, \&c.$  denote the roots of an equation of  $n-1$  dimensions  $v^{n-1} - pv^{n-2} + qv^{n-3} - \&c. = 0$ . It is evident, that in this case the equation whose root is  $x$  will have  $n^{n-1}$  dimensions; for let the roots of the equation  $z^n - 1 = 0$  be denoted by  $\alpha, \beta, \gamma, \delta, \&c.$  then will the quantity  $\sqrt[n]{\pi}$  have the  $n$  following values

values  $\alpha\sqrt[n]{\pi}$ ,  $\beta\sqrt[n]{\pi}$ ,  $\gamma\sqrt[n]{\pi}$ , &c. and the same may be affirmed of the quantities  $\sqrt[n]{\rho}$ ,  $\sqrt[n]{\sigma}$ ,  $\sqrt[n]{\tau}$ , &c. and consequently the quantity  $\sqrt[n]{\pi} + \sqrt[n]{\rho}$  will have  $n \times n$  different values; and in the same manner the root  $x = \sqrt[n]{\pi} + \sqrt[n]{\rho} + \sqrt[n]{\sigma} + \sqrt[n]{\tau} +$  may be proved to contain  $n \times n \times n \times n \times$  &c.  $= n^{n-1}$  roots, and consequently in this resolution, in equations of superior dimensions, the number of independent coefficients  $(n-1)$  will be very few in proportion to the number of dimensions  $n^{n-1}$ , or (if we respect its formula)  $n^{n-2}$  of the resulting equation.

Let  $n=3$ , and the equation resulting will rise to an equation of nine dimensions, which has the formula of a cubic; for let  $x = \sqrt[3]{\pi} + \sqrt[3]{\rho} = a$  one root, then will  $\frac{-1 + \sqrt{-3}}{2} a$  &  $\frac{-1 - \sqrt{-3}}{2} a$  be two other of the nine roots, and consequently the roots will be  $\overline{x^3 - a^3} \times \overline{x^3 - b^3} \times \overline{x^3 - c^3} = 0$ , which has the formula of a cubic: and in general the above-mentioned equation of  $n^{n-1}$  dimensions will, for the same reason, have the formula of an equation of  $n^{n-2}$  dimensions.

Let the resolution be  $x = \sqrt[n]{\pi} + \sqrt[n]{\rho} + \sqrt[n]{\sigma} + \sqrt[n]{\tau} +$  &c. where  $\pi, \rho, \sigma, \tau$ , &c. denote the roots of an equation  $x^{n-1} - px^{n-2} + qx^{n-3} -$  &c.  $= 0$  of  $(n-1)$  dimensions, then will the resulting equation free from radicals, whose root is  $x$ , rise to  $2^{n-1}$  dimensions; but as every affirmative

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tive has a negative root equal to it, it will have the formula of an equation of  $2^{n-2}$  dimensions.

Let the resolution be of this formula  $x = \sqrt[r \times m]{\alpha} + \sqrt[r \times m]{\beta} + \sqrt[r \times m]{\gamma} + \sqrt[r \times m]{\delta} + \&c.$  if  $\alpha, \beta, \gamma, \delta, \&c.$  be considered as the  $r$  power of the roots of an equation of  $s$  dimensions, then will the resulting equation, of which the resolution is given, rise only to an equation of the formula of  $m^{s-1}$  dimensions.

In the year 1762 I published some reasons, for which this method could not extend to the general resolution of algebraical equations.

